Magnetic Reconnection in Three Dimensional Space

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(1) For $\eta_{||}$ sufficiently small, a magnetic field undergoing a generic evolution will transition after a trigger time, $\tau_{trig} \propto 1/||\vec{\nabla}\vec{u}||$, into a state in which reconnection proceeds at an Alfvénic rate no matter how simple the initial state; $\vec{u}$ is the evolution velocity of the magnetic field lines. Only c/ω_{pe} need be non-zero. Effect absent in two-coordinate models.

(2) Fast reconnection relaxes $\vec{\nabla}(j_{||}/B)$ but conserves magnetic helicity since $< \vec{E}\cdot\vec{B} >= 0$. This releases a definite amount of magnetic energy in an Alfvén time into Alfvén waves, which are damped on the plasma (Similon-Sudan effect). $< \vec{E}\cdot\vec{B} >= 0$ makes particle acceleration subtle.

(3) For a dominant guide field (reduced MHD), $j_{||}/B$ has an evolution equation of the same form as the well-known equation for exponentially enhanced mixing by large-scale stirring in fluids. Only two spatial coordinates are required for enhanced fluid mixing, but three for exponentially enhanced reconnection.

What Prevents Magnetic Field Lines from Changing Connections

Where $\vec{B} \neq 0$, the electric field can always be written (Newcomb 1958)

$$\vec{E} + \vec{u} \times \vec{B} = -\vec{\nabla}\Phi, \text{ where } \frac{d\Phi}{d\ell} = -E_{||}.$$

In Clebsch coordinates $\vec{B} = \vec{\nabla}\alpha \times \vec{\nabla}\beta$*; the magnetic field line velocity* $\vec{u}$ *is defined by* $d\alpha/dt = \partial\alpha/\partial t + \vec{u} \cdot \vec{\nabla}\alpha = 0$*; differential distance along* $\vec{B}$ *is* $d\ell$*.*

When a well-behaved Φ exists, field lines do not break, and

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}). \qquad \text{Let } \vec{u} \cdot \vec{B} = 0.$$

The speed and prevalence of fast (Alfvénic) reconnection implies the cause is in this ideal evolution equation.

A credible theory of low-dissipation reconnection requires knowledge:

(1) Of the two boundary conditions on Φ.

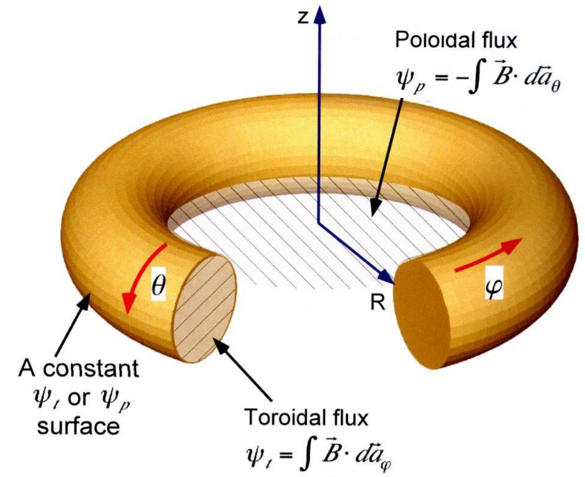
Changes in $\vec{B}$ *propagate through Alfvén waves.*

(2) Of the drive for the evolution of $\vec{B}$.

Magnetic Field Line Velocity $\vec{u}$

Outside of toroidal plasma physics, it is often denied that magnetic field lines have a velocity $\vec{u}$ that is distinct from the velocity of the plasma in which they are embedded.

What is actually meant by the plasma velocity $\vec{v}$ in a multispecies (at least electrons and ions)low collisionality plasma is unclear.



The magnetic field line velocity depends on the boundary conditions on Φ, where $E_{||} = -d\Phi/d\ell$. Without boundary conditions Φ can be chosen freely across the lines, which changes $\vec{u}$. Magnetic field lines are given by a Hamitonian, $\psi_p(\psi_t, \theta, \varphi)$, and changing Φ across the lines is a canonical transformation.

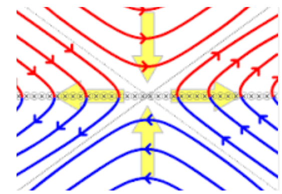
Can be understood in toroidal plasmas with the classical Ohm's law $\vec{E} + \vec{v} \times \vec{B} = \eta_{||}\vec{j}_{||} + \eta_\perp \vec{j}_\perp$, which can be rewritten $\vec{E} + \vec{u} \times \vec{B} = \eta_{||}\vec{j}_{||}$, where $\vec{u} = \vec{v} + \eta_\perp \vec{j}_\perp \times \vec{B}/B^2$.

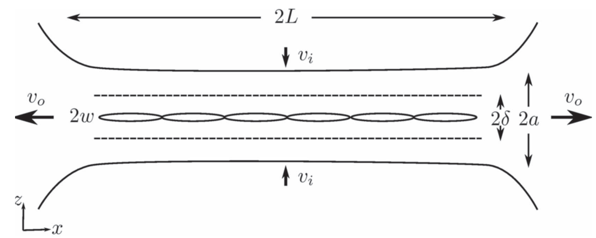
$\eta_{||}$ controls slippage of the poloidal ψ_p relative to the toroidal ψ_t magnetic flux. $\eta_\perp$ controls plasma flow across magnetic surfaces. Empirically $\eta_\perp/\eta_{||} \sim 10^5$.

Unrealism of two-coordinate X-point reconnection

(1) Fast reconnection during tokamak current spikes:

a. Inconsistent with 2D plasmoids.





Huang, Comisso, & Bhattacharjee, Ap.J. 849:75, 2017

Two-coordinate, plasmoid theories could greatly enhance reconnection near a single rational surface but not over a large range of safety factors. In JET, NF **56**, 026007 (2016), islands grow at a Rutherford-like rate at a number of rational surfaces on a time scale of 100's ms but then have a large-scale reconnection on less than 1 ms.

b. Prove nulls of $\vec{B}$ are not required for fast magnetic reconnection.

(2) Strong guide-field theory of reconnection:

Shows a large $j_{||}/B$ not needed and an unlikely explanation for fast reconnection.

$j_{||}/B$ obeys same differential equation in an evolving magnetic field in three, but not two, dimensions as does the mixing of cream in coffee by stirring. Exponential enhancement of mixing does not occur by concentrating the cream. The source term in $j_{||}/B$ equation is $d\Omega/d\ell$, where $\Omega = \hat{b}\cdot\vec{\nabla}\times\vec{u}$.

Although $\sim$25 papers are published each year on plasmoid reconnection, it clearly does not provide a basis for a general theory of fast reconnection but may explain some features using two-coordinate models. Concepts explained here are just in my papers.

Standard mathematics coupled with Maxwell's equations does.

- Why a magnetic field subject to a generic evolution will reach a state of fast (Alfvenic) reconnection within a time scale set by $1/||\vec{\nabla}\vec{u}||$.

- Why the reconnection process, which flattens $\vec{\nabla}(j_{||}/B)$, proceeds at the shear Alfvén speed, $V_A = B/\sqrt{\mu_0\rho_0}$.

- Why the helicity contained in the reconnecting region is conserved,
$$\int \vec{A}\cdot\vec{B}d^3x = constant.$$

- Why the energy released from the magnetic field has a definite value.

Langrangian coordinates for the magnetic field lines $\vec{x}_0$

$$\frac{d\vec{x}(\vec{x}_0,t)}{dt} \equiv \vec{u}(\vec{x},t), \text{ with } \vec{x}(\vec{x}_0,t=0)=\vec{x}_0.$$

The ideal evolution equation implies (Reviewed by Stern, Space Science Reviews **6**, 147 (1966).)

$$\vec{B}(\vec{x},t) = \frac{\overleftrightarrow{\mathcal{J}}}{\mathcal{J}} \cdot \vec{B}_0(\vec{x}_0), \text{ where}$$

$$\overleftrightarrow{\mathcal{J}} \equiv \frac{\partial \vec{x}}{\partial \vec{x}_0} \text{ is the Jacobian matrix and } \mathcal{J} \equiv \left\| \overleftrightarrow{\mathcal{J}} \right\| \text{ is the Jacobian.}$$

$$\overleftrightarrow{\mathcal{J}} \equiv \frac{\partial \vec{x}}{\partial \vec{x}_0} = \begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{pmatrix} = \overleftrightarrow{U} \cdot \begin{pmatrix} e^{\sigma_1} & 0 & 0 \\ 0 & e^{\sigma_2} & 0 \\ 0 & 0 & e^{\sigma_3} \end{pmatrix} \cdot \overleftrightarrow{V}^{\dagger}.$$

The finite time Lyapunov exponents of the flow are σ_1, σ_2, and σ_3.

One of the three finite-time Lyapunov exponents is zero

Since $\vec{u}\cdot\vec{B}=0$, magnetic flux conservation implies $\mathcal{J}B=B_0$, dotting $\vec{B}$ with itself gives

$$\left(\frac{\mathcal{J}B}{B_0}\right)^2=\hat{b}_0^\dagger\cdot\overleftrightarrow{V}\cdot\begin{pmatrix}e^{2\sigma_1}&0&0\\0&e^{2\sigma_2}&0\\0&0&e^{2\sigma_3}\end{pmatrix}\cdot\overleftrightarrow{V}^\dagger\cdot\hat{b}_0=1\text{ where }\hat{b}_0\equiv\vec{B}_0/B_0.$$

One singular value, taken to be e^{σ_3}, must be unity with $\hat{b}_t\equiv\overleftrightarrow{V}^\dagger\cdot\hat{b}_0$ the associated eigenvector. Consequently,

$$\overleftrightarrow{\mathcal{J}}=\overleftrightarrow{U}\cdot\begin{pmatrix}e^{\sigma_1}&0&0\\0&e^{\sigma_2}&0\\0&0&1\end{pmatrix}\cdot\overleftrightarrow{V}^\dagger,\text{ which implies }\vec{B}=\overleftrightarrow{U}\cdot\begin{pmatrix}e^{-\sigma_2}&0&0\\0&e^{-\sigma_1}&0\\0&0&e^{-(\sigma_1+\sigma_2)}\end{pmatrix}\cdot\overleftrightarrow{V}^\dagger\cdot\vec{B}_0;$$

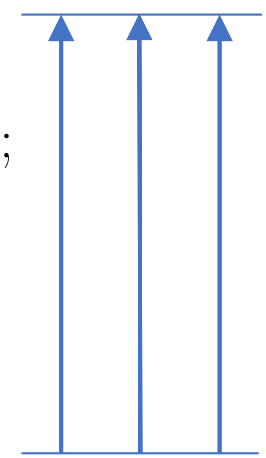
$$B=\frac{B_0}{\mathcal{J}}=B_0e^{-(\sigma_1+\sigma_2)}.$$

B can change, but only exponentially, which is not consistent with the formation of a null where none existed before. When the magnetic field strength does not exponentiate during a 3D evolution, there is only one Lyapunov exponent σ.

Evolution equation for $K \equiv j_{||}/B$ in reduced MHD (strong guide field)

When the magnetic field extends from a rigid perfect conductor at $z=0$ to a flowing, $\vec{u}_b = \vec{\nabla}\phi_b(x,y,t)\times\hat{z}$, perfect conductor at $z=L$:

Perfect flowing conductor



Perfect fixed conductor

$\vec{x} = z\hat{z} + \vec{x}_\perp(x_0,y_0,z,t)$ where

$$\vec{u} \equiv \frac{d\vec{x}_\perp(x_0,y_0,z,t)}{dt} \text{ is the field line velocity; } \vec{x}_\perp(x_0,y_0,z=0,t)\equiv x\hat{x}+y\hat{y};$$

$$\vec{B} = B_g(\hat{z}+\vec{\nabla}H(x,y,z,t)\times\hat{z}), \text{ so } \frac{dx}{dz} = \frac{\partial H}{\partial y} \text{ and } \frac{dy}{dz} = -\frac{\partial H}{\partial x}.$$

Evolution equation for $K \equiv (\hat{z}\cdot\vec{\nabla}\times\vec{B})/B$ is

$$\frac{\partial K}{\partial t} + \vec{u}\cdot\vec{\nabla}K - \frac{\eta}{\mu_0}\nabla_\perp^2 K = \left(\frac{d\Omega}{dz}\right)_{x_0 y_0}, \text{ where } \Omega = \hat{z}\cdot\vec{\nabla}\times\vec{u}.$$

In Lagrangian $x_0\hat{x}+y_0\hat{y}$ coordinates

$$\left(\frac{\partial K}{\partial t}\right)_{x_0,y_0} - \frac{\eta}{\mu_0}\sum_{i,j=1}^{2}\left(\frac{\partial}{\partial x_0^i} g^{ij}\frac{\partial K}{\partial x_0^j}\right) = \left(\frac{d\Omega}{dz}\right)_{x_0 y_0}. \text{ Note } \left(\frac{d\Omega}{dz}\right)_{x_0 y_0} = \frac{d\Omega}{d\ell}.$$

$$g^{ij} \equiv \left[\begin{pmatrix} \partial x/\partial x_0 & \partial y/\partial x_0 \\ \partial x/\partial y_0 & \partial y/\partial y_0 \end{pmatrix}\cdot\begin{pmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 \end{pmatrix}\right]^{-1} = \overset{\leftrightarrow}{V}\cdot\begin{pmatrix} e^{2\sigma} & 0 \\ 0 & e^{-2\sigma}\end{pmatrix}\cdot\overset{\leftrightarrow}{V}^{\dagger}.$$

Evolution of an impurity of density n in a stirred fluid

$$\frac{\partial n}{\partial t} + \vec{v}\cdot\vec{\nabla} n - D\nabla_\perp^2 n = S;\ \text{Impurity source is } S.$$

When $\vec{\nabla}\cdot\vec{v} = 0$ and 2D, the velocity is $\vec{v} = \vec{\nabla}\phi_f(x,y,t)\times\hat{z}$.

Streamlines obey $\dfrac{dx}{dt} = \dfrac{\partial\phi_f}{\partial y}$ and $\dfrac{dy}{dt} = -\dfrac{\partial\phi_f}{\partial x}$.

Lagrangian coordinates: $\dfrac{d\vec{x}(x_0,y_0,t)}{dt} = \vec{v}$

In Lagrangian coordinates $\left(\dfrac{\partial n}{\partial t}\right)_{x_0,y_0} - D\sum_{i,j=1}^{2}\left(\dfrac{\partial}{\partial x_0^i} g^{ij}\dfrac{\partial n}{\partial x_0^j}\right) = S.$

$$g^{ij} \equiv \left[\begin{pmatrix}\partial x/\partial x_0 & \partial y/\partial x_0\\ \partial x/\partial y_0 & \partial y/\partial y_0\end{pmatrix}\cdot\begin{pmatrix}\partial x/\partial x_0 & \partial x/\partial y_0\\ \partial y/\partial x_0 & \partial y/\partial y_0\end{pmatrix}\right]^{-1} = \overset{\leftrightarrow}{V}\cdot\begin{pmatrix}e^{2\sigma} & 0\\ 0 & e^{-2\sigma}\end{pmatrix}\cdot\overset{\leftrightarrow}{V}^{\dagger}.$$

Though having only two spatial dimensions, this equation can give exponential enhancement of mixing through a large-scale stirring velocity $\vec{v}_f$, Aref et al, RMP **89**, 025007 (2017).The evolution equation for K and n are the same, but three spatial dimensions are required for the K equation.

Why 2D adequate for fast mixing but 3D needed for prevalent fast magnetic reconnection

Exponential separation of trajectories of a Hamiltonian, $H(p, q, t)$, with pair of canonically conjugate coordinates requires a dependence on the canonical time.

When $\partial H/\partial t = 0$, H is a constant of the motion, trajectories follow constant-H contours, and do not generically exponentiate apart,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial p}\frac{dp}{dt} + \frac{\partial H}{\partial q}\frac{dq}{dt} = \frac{\partial H}{\partial t} \text{ because } \frac{\partial H}{\partial p} = \frac{dq}{dt} \text{ and } \frac{\partial H}{\partial q} = -\frac{dp}{dt}.$$

The Hamiltonian for streamlines in two dimensions $\phi_f(x, y, t)$ has the form needed for an exponential separation of trajectories.

Magnetic field lines are calculated at a fixed time. Their Hamiltonian is $H(x, y, z)$ with the spatial coordinate z the canonical time. All three spatial coordinates are required to obtain generic separation of magnetic field lines.

What is the trigger for fast magnetic reconnection

The rate at which a magnetic field approaches a reconnecting state is not Alfvénic; it is determined by flow speed $\vec{u}$; the Lyapunov exponent σ must become sufficiently large to trigger a reconnection.

In the reduced MHD model, the Lyapunov exponent $\sigma \sim \Omega_b t \sim (j_{||}/B)L$, where $\Omega_b = \hat{z} \cdot \vec{\nabla} \times \vec{u}_b$ is the driven vorticity and L is distance between boundaries.

The current density need not be large for a fast magnetic reconnection.
Required $j_{||}/B$ is proportional to the required σ.

In simulations, reconnection occurs where σ is large, Yi-Min Huang et al (2014) and W. Daughton et al (2014), but maximum σ that can be resolved in these codes is six to eight. For coronal simulations need $\sigma \sim 20$.

Why is magnetic reconnection Alfvénic

Can write $\vec{j} = (j_{||}/B)\vec{B} + \vec{B} \times (\vec{j} \times \vec{B})/B^2$, so $\vec{\nabla} \cdot \vec{j} = 0$ implies

$$\vec{B} \cdot \vec{\nabla}\frac{j_{||}}{B} = \vec{B} \cdot \vec{\nabla} \times \frac{\vec{f}}{B^2}, \text{ where } \vec{f} = \vec{j} \times \vec{B}$$

is the electromagnetic or Lorentz force. When $\vec{f} = \rho_0 d\vec{v}/dt$, relaxation of $j_{||}/B$ along the magnetic field is by Alfvén waves, $V_A = B/\sqrt{\mu_0\rho_0}$.

The reduced MHD equations for plasmas driven by moving boundaries give an Alfvén wave equation with dissipation when the plasma viscosity or resistivity is non-zero.

The Alfvén waves that relax $j_{||}/B$ travel along the magnetic field lines, so 100's of toroidal circuits can be required to relax $j_{||}/B$ during a tokamak thermal quench. Damping is enhanced by Similon-Sudan effect, Astrophys. J. **336**, 442 (1989).

An important issue in tokamak disruption physics is the spreading of impurities by magnetic reconnection.

Why is magnetic helicity conserved

Fast magnetic reconnection is a quasi-ideal process that conserves magnetic helicity.

When field lines don't obey periodicity constraints of magnetic surfaces, $\vec{E}+\vec{u}\times\vec{B}=-\vec{\nabla}\Phi$ implies

$$\int \vec{E}\cdot\vec{B}d^3x=-\oint \Phi\vec{B}\cdot d\vec{a}=0$$

since normal field is zero on the boundaries of stochastic regions.

$$\frac{\partial}{\partial t}\int \vec{A}\cdot\vec{B}d^3x=-2\int \vec{E}\cdot\vec{B}d^3x-\oint(\vec{E}\times\vec{A}+\Phi_g\vec{B})\cdot d\vec{a},\ \text{where } \vec{E}=-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\Phi_g$$

When $\vec{E}+\vec{u}\times\vec{B}=-\vec{\nabla}\Phi$, helicity flux $\vec{E}\times\vec{A}=\vec{u}\vec{A}\cdot\vec{B}+\Phi\vec{B}-\vec{\nabla}\times(\Phi\vec{A})$

Importance to current transfer to relativistic electrons in tokamaks:

(a) Large changes in $\psi_p(\psi_t)$ cannot directly accelerate electrons when the current profile relaxes during a thermal quench, $V_\ell=(\partial\psi_p/\partial t)_{\psi_t}$,

(b) but the relaxation can produce surface currents that can.

Summary

An ideal (dissipationless) evolution of a magnetic field in general leads to a state in which the magnetic field lines change their connections on an Alfvénic (inertial) time scale. *Requires only a finite mass for the lightest current carrier, the electron.*

During a fast magnetic reconnection, $\vec{\nabla}(j_{||}/B)$ relaxes while conserving magnetic helicity in the reconnecting region. *This implies a definite amount of energy is released from the magnetic field and transferred to shear Alfvén waves, which in turn transfer their energy to the plasma.*

When there is a strong non-reconnecting component of the magnetic field, called a guide field, $j_{||}/B$ obeys the same evolution equation as that of an impurity being mixed into a fluid by stirring. *Although the enhancement of mixing by stirring has been recognized by every cook for many millennia, the analogous effect in magnetic reconnection is not generally recognized.*

Interestingly, a three-coordinate model is required for the enhancement of magnetic reconnection while only two coordinates are required in fluid mixing. *The issue is the number of spatial coordinates required to obtain an exponentiating spatial separation of magnetic field lines versus streamlines of a fluid flow.*